Special Topic :
NP-hard, Reduction

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prerequisite

know graph

know DP

motivation problem

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given a graph, each node has a weight.

you want to choose subset of nodes with maximum total weight. any pair of chosen nodes must not be adjacent.

some definitions

P =

can be solved in polynomial time

NP =
can be checked in
polynomial time

NP-hard =
if you can solve this in poly time, you can solve all problems in NP in poly time

no one has found polynomial solution to any NP-hard problem

research since 1971, unlikely to be solved in 5 hours

optimisation vs decision problem

decision problem

given N vertices, **can you** choose at most K vertices s.t. for each edge (a,b), at least one of the vertices is chosen?

optimisation problem

given N vertices, **find minimum** number of vertices to be chosen for each edge (a,b), at least one of the vertices is chosen?

decision <=> optimisation

why?

methods to know whether a problem is NP-hard

reduction

notation + definition 1 :

Y polynomial-time reduce to X (notation Y ≤p X) <=> if you can solve X in polynomial time, then you can solve Y in polynomial time

in other words

$Y \leq_p X$ <=> you can “use” X to solve Y suppose you want to know whether problem X is NP-hard if you can find an NP-hard problem Y, and Y $\leq_p$ X, then X is NP-hard

by contradiction

let’s begin the first problem

3-SAT

given a conjunction of several clauses, where each clause is disjunction of 3 literals.
find whether the conjuction is satisfiable

example

(a OR -a OR -b) AND
(c OR b OR d) AND
(-a OR -c OR -d)

example

(a OR b OR c) AND
(a OR b OR -c) AND
(a OR -b OR c) AND
(a OR -b OR -c) AND
(-a OR b OR c) AND
(-a OR b OR -c) AND
(-a OR -b OR c) AND
(-a OR -b OR -c)

for now, let's

accept without proof, that

$$3\text{-SAT} \in \text{NP-hard}$$

now, tasks

MAX-CLIQUE

we prove that MAX-CLIQUE is NP-hard

3-SAT $\leq_p$ MAX-CLIQUE

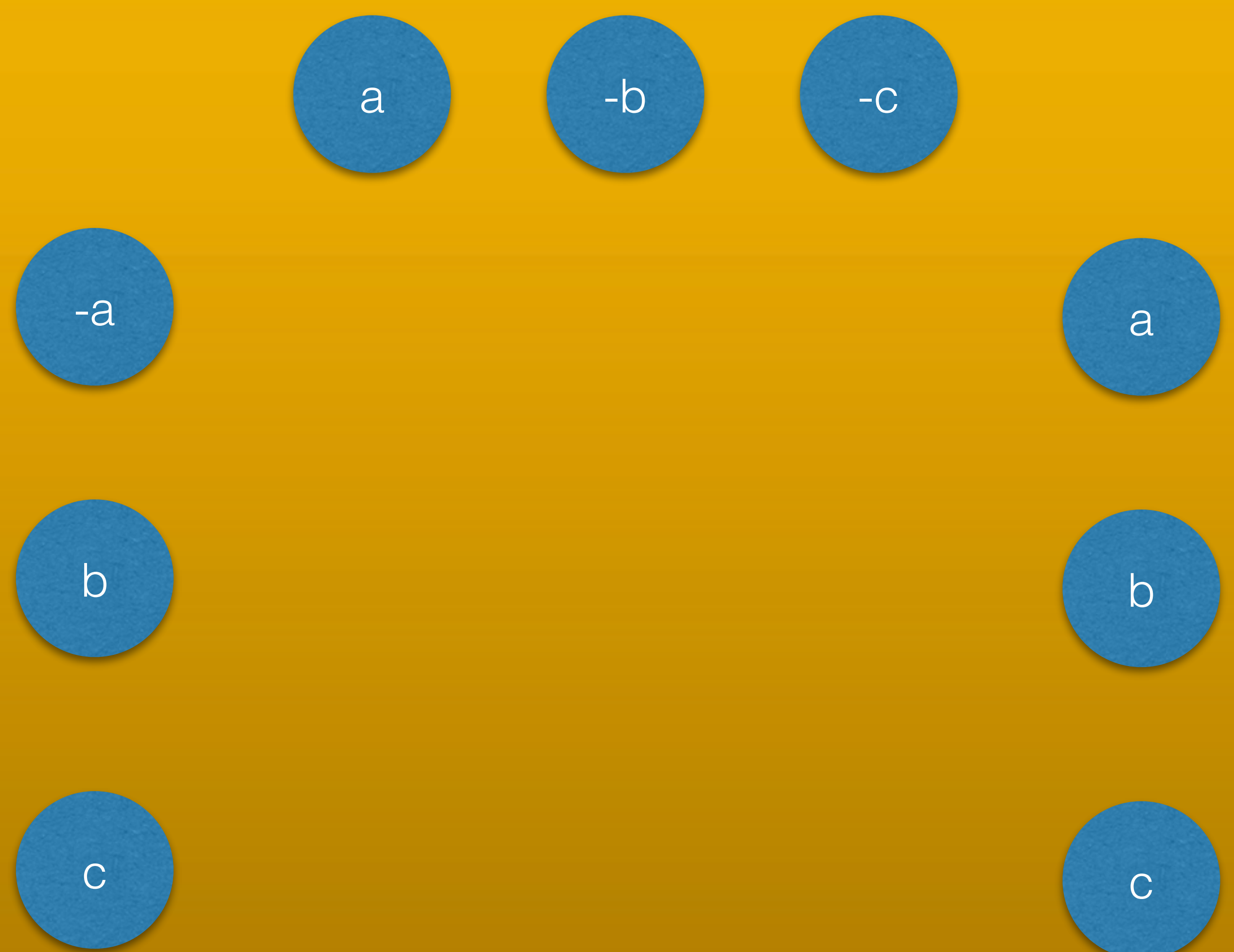
example :

(a v -b v -c) ^ (-a v b v c) ^ (a v b v c)

example :

(a v -b v -c) ^ (-a v b v c) ^ (a v b v c)

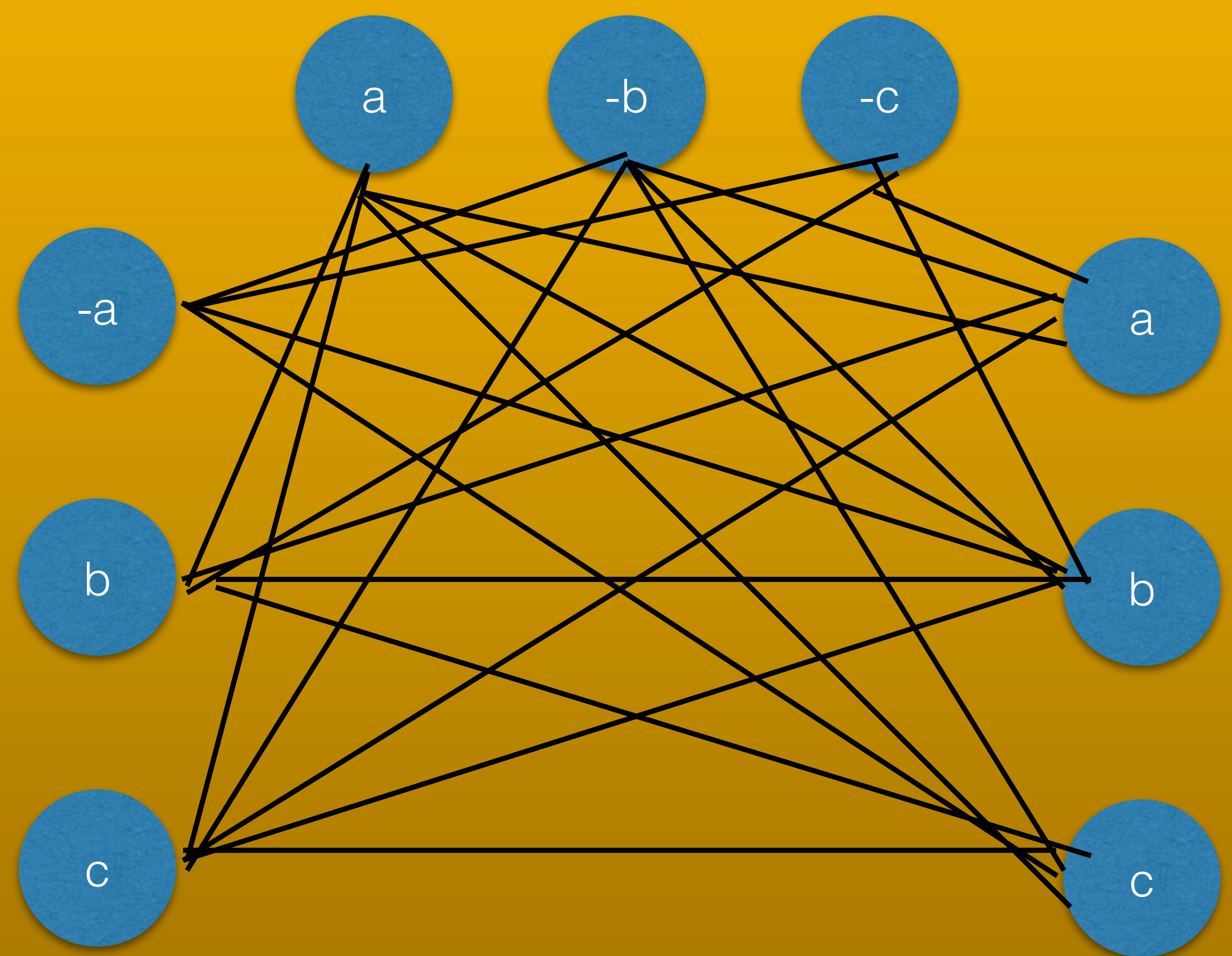
we create a node for each literal



example :

(a v -b v -c) ^ (-a v b v c) ^ (a v b v c)

for each node (x,y) we add an edge iff
(1) they are from a different clause, and
(2) x is not a negation of y



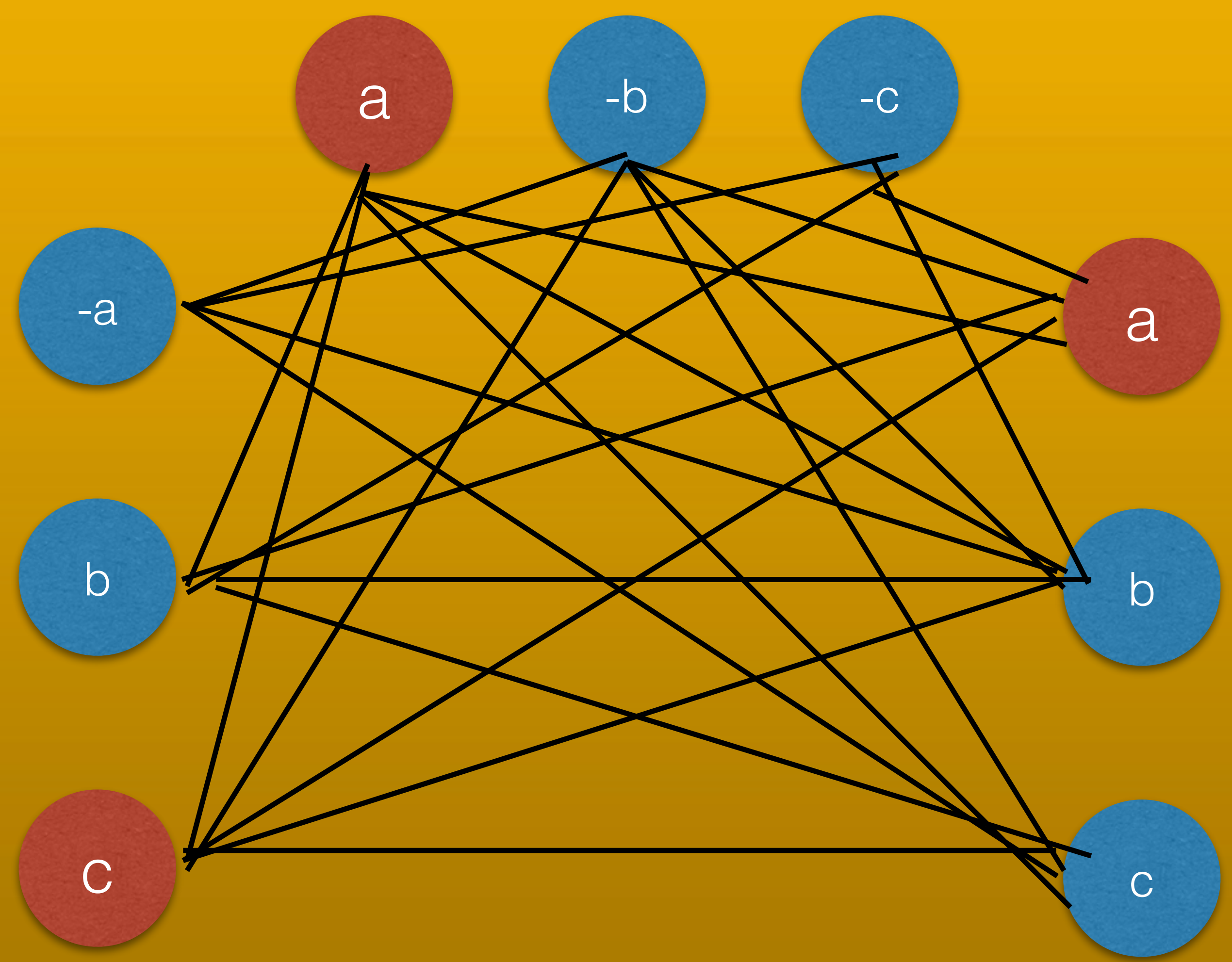
example :

(a v -b v -c) ^ (-a v b v c) ^ (a v b v c)

MAX-CLIQUE ≥ number of clauses

<=>

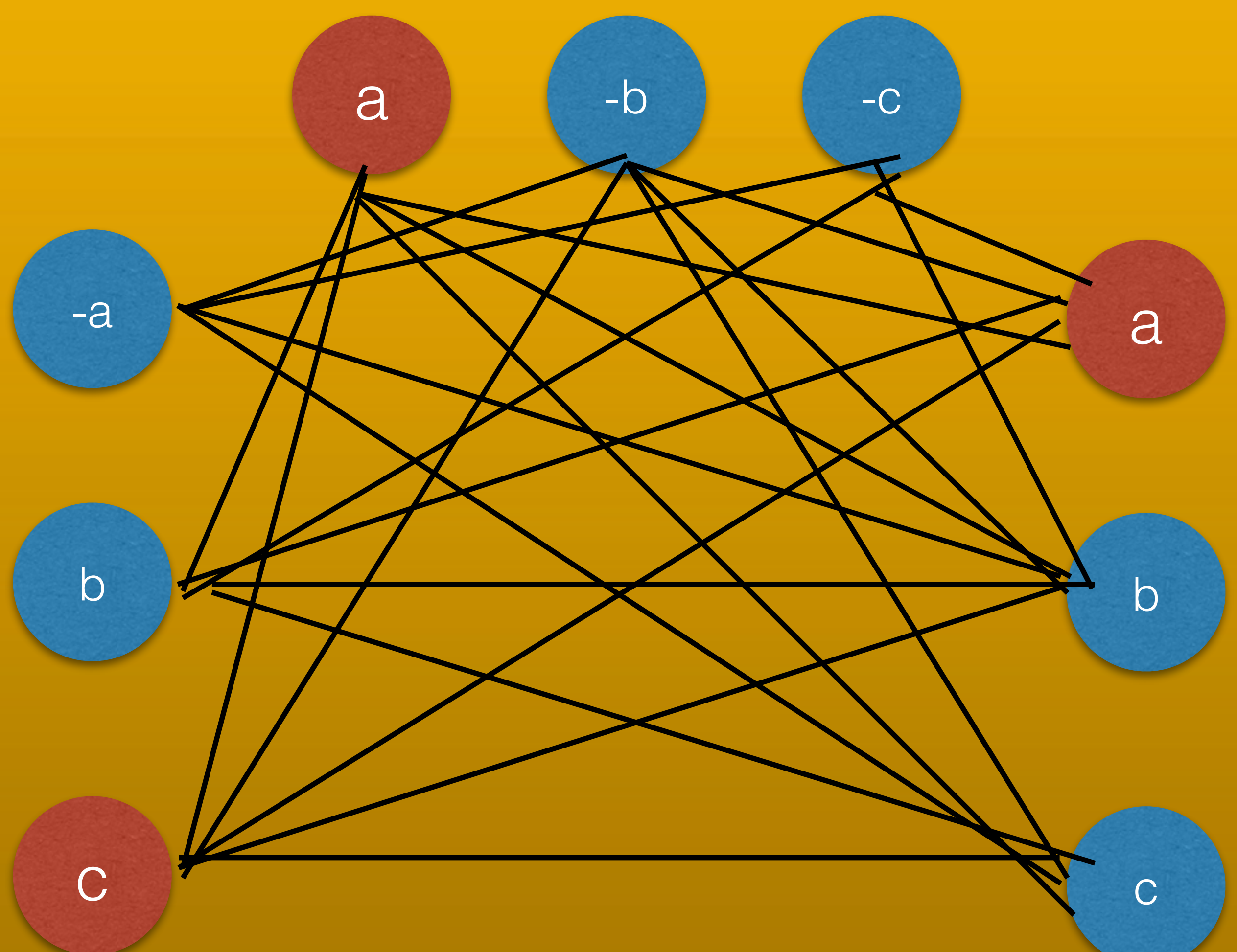
3-SAT is satisfiable



example :

(a v -b v -c) ^ (-a v b v c) ^ (a v b v c)

chosen node in CLIQUE <=> the true literal



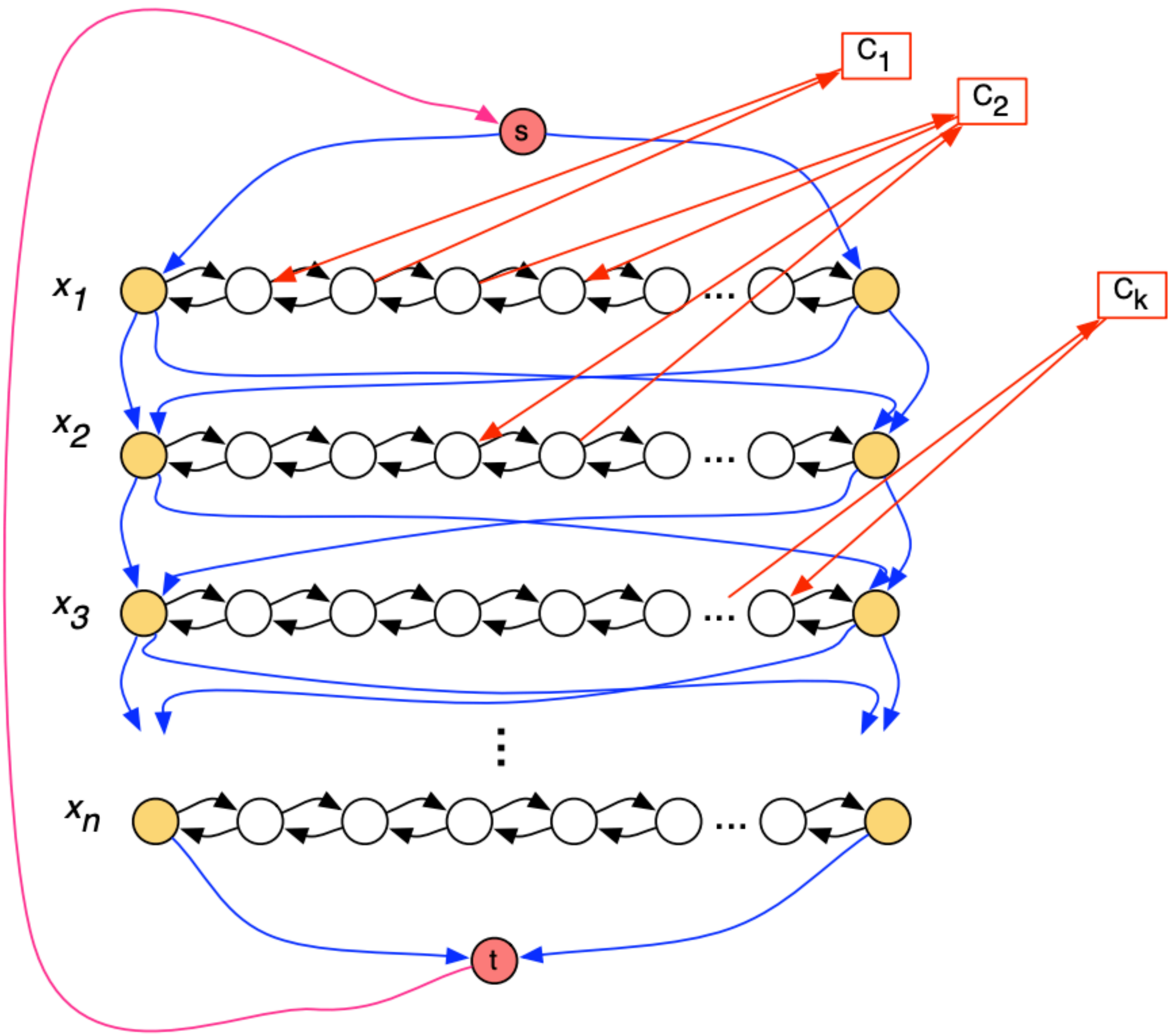
proven :

if we can solve MAX-CLIQUE in polynomial time, we can solve 3-SAT in polynomial time

since 3-SAT NP-hard,
MAX-CLIQUE is also NP-hard

another reduction

3-SAT $\leq_p$ HAMILTONIAN-PATH



C_1
C_2
C_k
s
x_1
x_2
x_3
x_n
t

another reduction

3-SAT ≤p TRIANGLE PARTITION

how do we proceed when encountering NP-hard problem?

tips 1: check constraint

SUBSET SUM

given an array N, find a subset that sums to K

SUBSET SUM is NP-hard

3-SAT $\leq_p$ NP-hard

Construction. Given 3-SAT instance Φ with n variables and k clauses, form 2n + 2k decimal integers, each of n+k digits, as illustrated below.

Claim. Φ is satisfiable iff there exists a subset that sums to W.

Pf. No carries possible.

$$C_1 = \bar{x} \lor y \lor z$$
$$C_2 = x \lor \bar{y} \lor z$$
$$C_3 = \bar{x} \lor \bar{y} \lor \bar{z}$$

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,110
¬ x	1	0	0	1	0	1	100,001
y	0	1	0	1	0	0	10,000
¬ y	0	1	0	0	1	1	10,111
z	0	0	1	1	1	0	1,010
¬ z	0	0	1	0	0	1	1,101
dummies to get clause columns to sum to 4	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

SUBSET SUM

given an array N of positive integers, find a subset that sums to K

$1 \leq N$, $\mathbf{K \leq 1000}$

tip 2: check for special property of the problem

given S = first N fibonaci number {1,1,2,3,…}
determine whether you can partition S to two equal sum subset

SUBSET-SUM $\leq_p$ PARTITION-SUM

SUBSET-SUM

given array A and find subset with total K

<=>

PARTITION-SUM

find partition in $A \cup \{K - (sum(A) - K)\}$

SUBSET-SUM

A : {1,2,**3**,4,**5**} K = 8

K - (1 + 2 + 3 + 4 + 5 - K) = 8 - (15 - 8) = 1

PARTITION-SUM

A : {1,2,**3**,4,**5**,1}

PARTITION-SUM itu NP-hard

so?

```
int main()
{
  int n;
  cin >> n;
  puts(n % 3 == 1 ? “no” : “yes”);
}
```

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given a graph, each node has a weight.

you want to choose subset of nodes with maximum total weight. any pair of chosen nodes must not be adjacent.

MAX INDEPENDENT SET

MAX CLIQUE ≤p MAX INDEPENDENT SET

the edges are added incrementally. for each i from 2 to N, given j ($1 \leq i < j$) the edges added are either:

1. edges connecting j to i
2. edges connecting all j neighbours to i
3. both 1 and 2

special graph :

1. satisfies triangle inequality
2. planar
3. bipartite

Google Code Jam
2008
Milkshakes

There are N milkshake flavors, each can be either prepared malted or not
There are M customers, each has a set of milkshakes that they like. At most one of them is malted. They will be happy if you have at least one of those type prepared.
Minimize the number of flavor that is malted to satisfy all customers

$1 \le N, M \le 2000$

EOF

Q&A?